HODGE-TATE AND DE RHAM REPRESENTATIONS IN THE IMPERFECT RESIDUE FIELD CASE

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Résumé. Soit $K$ un corps local $p$-adique de corps résiduel $k$ tel que $[k : k^p] = p^e < +\infty$ et soit $V$ une représentation $p$-adique de $\text{Gal}(\overline{K}/K)$. Nous utilisons la théorie des modules différentiels $p$-adiques pour montrer que $V$ est une représentation de Hodge-Tate (resp. de Rham) de $\text{Gal}(\overline{K}/K)$ si et seulement si $V$ est une représentation de Hodge-Tate (resp. de Rham) de $\text{Gal}(\overline{K}^{pf}/K^{pf})$ où $K^{pf}/K$ est un certain corps local $p$-adique de corps résiduel le plus petit corps parfait $k^{pf}$ contenant $k$.

Abstract. Let $K$ be a $p$-adic local field with residue field $k$ such that $[k : k^p] = p^e < +\infty$ and $V$ be a $p$-adic representation of $\text{Gal}(\overline{K}/K)$. Then, by using the theory of $p$-adic differential modules, we show that $V$ is a Hodge-Tate (resp. de Rham) representation of $\text{Gal}(\overline{K}/K)$ if and only if $V$ is a Hodge-Tate (resp. de Rham) representation of $\text{Gal}(\overline{K}^{pf}/K^{pf})$ where $K^{pf}/K$ is a certain $p$-adic local field with residue field the smallest perfect field $k^{pf}$ containing $k$.

1. Introduction

Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Choose an algebraic closure $\overline{K}$ of $K$ and put $G_K = \text{Gal}(\overline{K}/K)$. By a $p$-adic representation of $G_K$, we mean a finite dimensional vector space $V$ over $\mathbb{Q}_p$ endowed with a continuous action of $G_K$. In the case $e = 0$ (i.e. $k$ is perfect), following Fontaine, we can classify $p$-adic representations of $G_K$ by using the $p$-adic periods rings $B_{\text{HT}}$, $B_{\text{dR}}$, $B_{\text{st}}$ and $B_{\text{cris}}$ (Hodge-Tate, de Rham, semi-stable and crystalline representations). In the general case (i.e. $k$ is not necessarily perfect), Hyodo constructed the imperfect residue field version of the ring $B_{\text{HT}}$ and Tsuzuki and several authors constructed that of the ring $B_{\text{dR}}$. By using these rings, we can define the imperfect residue field version of Hodge-Tate and de Rham representations of $G_K$ in the evident way ([Br2],[H],[K1],[K2],[Tz]).

Now, we shall state the main result of this article. Let us fix some notations. Fix a lifting $(b_i)_{1 \leq i \leq e}$ of a $p$-basis of $k$ in $\mathcal{O}_K$ (the ring of integers of $K$) and for

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each \( m \geq 1 \), fix a \( p^m \)-th root \( b_1^{1/p^m} \) of \( b_1 \) in \( K \) satisfying \( (b_1^{1/p^m+1})^p = b_1^{1/p^m} \). Put \( K_{p^f} = \bigcup_{m \geq 1} K(b_1^{1/p^m}, 1 \leq i \leq e) \) and \( K_{pf} \) the \( p \)-adic completion of \( K_{p^f} \). These fields depend on the choice of a lifting of a \( p \)-basis of \( k \) in \( \mathcal{O}_K \). Since \( K_{pf} \) becomes a complete discrete valuation field with perfect residue field, we can apply theories in the perfect residue field case to \( p \)-adic representations of \( G_{K_{pf}} = \text{Gal}(K_{pf}/K_{pf}) \) where we choose an algebraic closure \( \overline{K_{pf}} \) of \( K_{pf} \) containing \( \overline{K} \). Note that, if \( V \) is a \( p \)-adic representation of \( G_K \), it can be also regarded as a \( p \)-adic representation of \( G_{K_{pf}} \) (see Section 2.2 for details). Our main result is the following.

**Theorem 1.1.** Let \( K \) be a complete discrete valuation field of characteristic 0 with residue field \( k \) of characteristic \( p > 0 \) such that \( [k : k^p] = p^e < +\infty \) and \( V \) be a \( p \)-adic representation of \( G_K \). Let \( K_{pf} \) be the field extension of \( K \) defined as above. Then, we have the following equivalences

1. \( V \) is a Hodge-Tate representation of \( G_K \) if and only if \( V \) is a Hodge-Tate representation of \( G_{K_{pf}} \).
2. \( V \) is a de Rham representation of \( G_K \) if and only if \( V \) is a de Rham representation of \( G_{K_{pf}} \).

In the case of Hodge-Tate representations, Tsuji \([Tj]\) had proved a more refined theorem based on this article. This paper is organized as follows. In Section 2, we shall review the definitions and basic known facts on Hodge-Tate and de Rham representations, first in the perfect residue field case and then in the imperfect residue field case. In Section 3, we shall review the theory of \( p \)-adic differential modules which play an central role in this article. In Section 4, by using the theory of \( p \)-adic differential modules, we shall prove the main theorem, first for Hodge-Tate representations and then for de Rham representations.

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2. **Preliminaries on Hodge-Tate and de Rham representations**

2.1. **Hodge-Tate and de Rham representations in the perfect residue field case.** (See \([F1]\) and \([F2]\) for details.) Let \( K \) be a complete discrete valuation field of characteristic 0 with perfect residue field \( k \) of characteristic \( p > 0 \). Choose an algebraic closure \( \overline{K} \) of \( K \) and consider its \( p \)-adic completion \( \mathbb{C}_p \). Put

\[
\overline{E} = \lim_{\leftarrow} \mathbb{C}_p = \{ (x^{(0)}, x^{(1)}, \ldots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p \}
\]
and let \( \wtilde{E}^+ \) denote the set of \( x = (x^{(i)}) \in \wtilde{E} \) such that \( x^{(0)} \in \mathcal{O}_p \) where \( \mathcal{O}_p \) denotes the ring of integers of \( \mathbb{C}_p \). For two elements \( x = (x^{(i)}) \) and \( y = (y^{(i)}) \) of \( \wtilde{E} \), their sum and product are defined by \( (x + y)^{(i)} = \lim_{j \to +\infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \) and \( (xy)^{(i)} = x^{(i)}y^{(i)} \). These sum and product make \( \wtilde{E} \) a perfect field of characteristic \( p > 0 \) (\( \wtilde{E}^+ \) is a subring of \( \wtilde{E} \)). Let \( \epsilon = (\epsilon^{(n)}) \) be an element of \( \wtilde{E} \) such that \( \epsilon^{(0)} = 1 \) and \( \epsilon^{(1)} \neq 1 \). Then, \( \wtilde{E} \) is the completion of an algebraic closure of \( k(\epsilon) \). The valuation defined by \( v_\wtilde{E}(x) = v_p(x^{(0)}) \) where \( v_p \) denotes the \( p \)-adic valuation of \( \mathbb{C}_p \) is normalized by \( v_p(p) = 1 \). The field \( \wtilde{E} \) is equipped with a continuous action of the Galois group \( G_K = \text{Gal}(\wtilde{K}/K) \) with respect to the topology defined by the valuation \( v_\wtilde{E} \). Put \( \wtilde{A}^+ = W(\wtilde{E}^+) \) (the ring of Witt vectors with coefficients in \( \wtilde{E}^+ \)) and \( \wtilde{B}^+ = \wtilde{A}^+[1/p] = \{ \sum_{k > -\infty} p^k x_k | x_k \in \wtilde{E}^+ \} \) where \([*]\) denotes the Teichmüller lift of \(* \in \wtilde{E}^+ \). This ring \( \wtilde{B}^+ \) is equipped with a surjective homomorphism

\[
\theta : \wtilde{B}^+ \to \mathbb{C}_p : \sum p^k x_k \mapsto \sum p^k x^{(0)}_k.
\]

If \( \tilde{p} = (p^{(n)}) \) denotes an element of \( \wtilde{E}^+ \) such that \( p^{(0)} = p \), we can show that \( \ker(\theta) \) is the principal ideal generated by \( \omega = [\tilde{p}] - p \). The ring \( B_{\text{dR},K}^+ \) is defined to be the \( \ker(\theta) \)-adic completion of \( \wtilde{B}^+ \)

\[
B_{\text{dR},K}^+ = \varprojlim_{n \geq 0} \wtilde{B}^+/(\ker(\theta)^n).
\]

This is a discrete valuation ring and \( t = \log([\epsilon]) \) which converges in \( B_{\text{dR},K}^+ \) is a generator of the maximal ideal. Put \( B_{\text{dR},K} = B_{\text{dR},K}^+[1/t] \). This ring \( B_{\text{dR},K} \) becomes a field and is equipped with an action of the Galois group \( G_K \) and a filtration defined by \( \text{Fil}^i B_{\text{dR},K} = t^i B_{\text{dR},K}^+ (i \in \mathbb{Z}) \). Then, \( (B_{\text{dR},K})^{G_K} \) is canonically isomorphic to \( K \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{dR},K}(V) = (B_{\text{dR},K} \otimes_{\mathbb{C}_p} V)^{G_K} \) is naturally a \( K \)-vector space. We say that a \( p \)-adic representation \( V \) of \( G_K \) is a de Rham representation of \( G_K \) if we have

\[
\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{dR},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{dR},K}(V)).
\]

Furthermore, we say that a \( p \)-adic representation \( V \) of \( G_K \) is a potentially de Rham representation of \( G_K \) if there exists a finite field extension \( L/K \) in \( \wtilde{K} \) such that \( V \) is a de Rham representation of \( G_L \). It is known that a potentially de Rham representation \( V \) of \( G_K \) is a de Rham representation of \( G_K \) (see [F2], 3.9).

Define \( B_{\text{HT},K} \) to be the associated graded algebra to the filtration \( \text{Fil}^i B_{\text{dR},K} \). The quotient \( \text{gr}^i B_{\text{HT},K} = \text{Fil}^i B_{\text{dR},K}/\text{Fil}^{i+1} B_{\text{dR},K} \) is a one-dimensional \( \mathbb{C}_p \)-vector space spanned by the image of \( t^i \). Thus, we obtain the presentation

\[
B_{\text{HT},K} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)
\]

where \( \mathbb{C}_p(i) = \mathbb{C}_p \otimes \mathbb{Z}_p(i) \) is the Tate twist. Then, \( (B_{\text{HT},K})^{G_K} \) is canonically isomorphic to \( K \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{HT},K}(V) = \)
(B_{HT,K} \otimes_{\mathbb{Q}_p} V)^{G_K} is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_K$ is a Hodge-Tate representation of $G_K$ if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{HT,K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{HT,K}(V)).$$

Furthermore, we say that a $p$-adic representation $V$ of $G_K$ is a potentially Hodge-Tate representation of $G_K$ if there exists a finite field extension $L/K$ in $\overline{K}$ such that $V$ is a Hodge-Tate representation of $G_L$. It is known that a potentially Hodge-Tate representation $V$ of $G_K$ is a Hodge-Tate representation of $G_K$ (see [F2], 3.9). Since we have $\gr B_{dR,K} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$, if $V$ is a de Rham representation of $G_K$, there exists a $G_K$-equivariant isomorphism $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} \mathbb{C}_p(n_j)$ $(n_j \in \mathbb{Z})$. Thus, it follows that a de Rham representation $V$ of $G_K$ is a Hodge-Tate representation of $G_K$.

### 2.2. Hodge-Tate and de Rham representations in the imperfect residue field case

Let $K$ be a complete discrete valuation field of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Choose an algebraic closure $\overline{K}$ of $K$ and put $G_K = \Gal(\overline{K}/K)$. As in Introduction, fix a lifting $(b_i)_{1 \leq i \leq e}$ of a $p$-basis of $k$ in $\mathcal{O}_K$ (the ring of integers of $K$) and for each $m \geq 1$, fix a $p^m$-th root $b_1^{1/p^m}$ of $b_1$ in $\overline{K}$ satisfying $(b_1^{1/p^m + 1})^p = b_1^{1/p^m}$. Put $K^{(pf)} = \bigcup_{m \geq 0} K(b_1^{1/p^m}, 1 \leq i \leq e)$ and $K^{pf} = \text{the } p\text{-adic completion of } K^{(pf)}$. These fields depend on the choice of a lifting of a $p$-basis of $k$ in $\mathcal{O}_K$. Since $K^{(pf)}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K^{pf}} = \Gal(\overline{K}^{pf}/K^{pf}) \simeq G_{K^{(pf)}} = \Gal(\overline{K}/K^{(pf)}) \subset G_K$ where we choose an algebraic closure $\overline{K}^{pf}$ of $K^{pf}$ containing $\overline{K}$. With this isomorphism, we identify $G_{K^{pf}}$ with a subgroup of $G_K$. We have a bijective map from the set of finite extensions of $K^{pf}$ contained in $\overline{K}$ to the set of finite extensions of $K^{pf}$ contained in $\overline{K}^{pf}$ defined by $L \to L\overline{K}^{pf}$. Furthermore, $L\overline{K}^{pf}$ is the $p$-adic completion of $L$. Hence, we have an isomorphism of rings

$$\mathcal{O}_{\overline{K}}/p^n \mathcal{O}_{\overline{K}} \simeq \mathcal{O}_{\overline{K}^{pf}}/p^n \mathcal{O}_{\overline{K}^{pf}}$$

where $\mathcal{O}_{\overline{K}}$ and $\mathcal{O}_{\overline{K}^{pf}}$ denote the rings of integers of $\overline{K}$ and $\overline{K}^{pf}$. Thus, the $p$-adic completion of $\overline{K}$ is isomorphic to the $p$-adic completion of $\overline{K}^{pf}$, which we will write $\mathcal{C}_p$. As in Subsection 2.1, construct the rings $\overline{E}^+$ and $\overline{A}^+ = W(\overline{E}^+)$ from this $\mathcal{C}_p$. Let $k^{pf}$ denote the perfect residue field of $K^{pf}$ and put $\mathcal{O}_{K_0} = \mathcal{O}_K \cap W(k^{pf})$. Let $\alpha : \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \overline{A}^+ \rightarrow \mathcal{O}_{K^{pf}}/p\mathcal{O}_{K^{pf}}$ be the natural surjection and define $\tilde{A}^+_{(K)}$ to be $\tilde{A}^+_{(K)} = \lim_{n \geq 0}(\mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \overline{A}^+)/((\Ker(\alpha))^n)$. Let $\theta_K : \tilde{A}^+_{(K)} \otimes_{\mathcal{Z}_p} \mathbb{Q}_p \rightarrow \mathcal{C}_p$ be the natural extension of $\theta : \overline{A}^+[1/p] \rightarrow \mathbb{C}_p$. Define $B^{+}_{dR,K}$ to be the $\Ker(\theta_K)$-adic completion of $\tilde{A}^+_{(K)} \otimes_{\mathcal{Z}_p} \mathbb{Q}_p$

$$B^{+}_{dR,K} = \lim_{n \geq 0}(\tilde{A}^+_{(K)} \otimes_{\mathcal{Z}_p} \mathbb{Q}_p)/((\Ker(\theta_K))^n).$$

This is a $K$-algebra equipped with an action of the Galois group $G_K$. Let $\tilde{b}_i$ denote $(b_i^{(m)}) \in \overline{E}^+$ such that $b_i^{(0)} = b_i$ and then the series which defines $\log([\tilde{b}_i]/b_i)$
converges to an element $t_i$ in $B_{\text{dr},K}^+$. Then, the ring $B_{\text{dr},K}^+$ becomes a local ring with the maximal ideal $m_{\text{dr}} = (t, t_1, \ldots, t_e)$. Define a filtration on $B_{\text{dr},K}^+$ by $\text{fil}^n B_{\text{dr},K}^+ = m_{\text{dr}}^n$. Then, the homomorphism

$$f : B_{\text{dr},K}^+[[t_1, \ldots, t_e]] \to B_{\text{dr},K}^+$$

is an isomorphism of filtered algebras (see [Br2], Proposition 2.9). From this isomorphism, it follows easily that

$$i : B_{\text{dr},K^p}^+ \hookrightarrow B_{\text{dr},K}^+ \quad \text{and} \quad p : B_{\text{dr},K}^+ \to B_{\text{dr},K^p}^+ : t_i \mapsto 0$$

are $G_{K^{p}}$-equivariant homomorphisms and the composition

$$p \circ i : B_{\text{dr},K^p}^+ \hookrightarrow B_{\text{dr},K}^+ \to B_{\text{dr},K^p}^+$$

is an identity. Put $B_{\text{dr},K} = B_{\text{dr},K}[1/t]$. Then, $K$ is canonically embedded in $B_{\text{dr},K}$ and we have a canonical isomorphism $(B_{\text{dr},K})^{G_K} = K$. Thus, for a $p$-adic representation $V$ of $G_K$, $D_{\text{dr},K}(V) = (B_{\text{dr},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_K$ is a de Rham representation of $G_K$ if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{dr},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{dr},K}(V)).$$

Furthermore, we say that a $p$-adic representation $V$ of $G_K$ is a potentially de Rham representation of $G_K$ if there exists a finite field extension $L/K$ in $\overline{K}$ such that $V$ is a de Rham representation of $G_L$. We can show that a potentially de Rham representation $V$ of $G_K$ is a de Rham representation of $G_K$ in the same way as in the perfect residue field case.

Define a filtration on $B_{\text{dr},K}$ to be

$$\text{Fil}^0 B_{\text{dr},K} = \sum_{n=0}^{\infty} t^{-n} \text{fil}^n B_{\text{dr},K}^+ = B_{\text{dr},K}[\frac{t_1}{t}, \ldots, \frac{t_e}{t}] ,$$

$$\text{Fil}^i B_{\text{dr},K} = t^i \text{Fil}^0 B_{\text{dr},K} \quad (i \in \mathbb{Z}).$$

Define $B_{\text{HT},K}$ to be the associated graded algebra to this filtration. Since the quotient $\text{gr}^i B_{\text{HT},K} = \text{Fil}^i B_{\text{dr},K}/\text{Fil}^{i+1} B_{\text{dr},K} \ (i \in \mathbb{Z})$ is given by $\text{gr}^i B_{\text{HT},K} = t^i \mathbb{C}_p[t_1, \ldots, \frac{t_e}{t}]$, we obtain the presentation

$$B_{\text{HT},K} = \mathbb{C}_p[t, t^{-1}, \frac{t_1}{t}, \ldots, \frac{t_e}{t}] = B_{\text{HT},K^p}[\frac{t_1}{t}, \ldots, \frac{t_e}{t}].$$

From this presentation, it follows easily that

$$i : B_{\text{HT},K^p} \hookrightarrow B_{\text{HT},K} \quad \text{and} \quad p : B_{\text{HT},K} \to B_{\text{HT},K^p} : t_i/t \mapsto 0$$

are $G_{K^{p}}$-equivariant homomorphisms and the composition

$$p \circ i : B_{\text{HT},K^p} \hookrightarrow B_{\text{HT},K} \to B_{\text{HT},K^p}$$

is an identity. The field $K$ is canonically embedded in $B_{\text{HT},K}$ and we have $(B_{\text{HT},K})^{G_K} = K$. Thus, for a $p$-adic representation $V$ of $G_K$, $D_{\text{HT},K}(V) = \ldots \ldots$
We shall review the definitions of a lifting of a characteristic $k$ plays an important role in this article. First, let us fix the notations. Let $H$ denote the kernel of the cyclotomic character $\chi : K_{/K} \to \mathbb{Z}_p^*$. Then, the Galois group $H$ is isomorphic to the subgroup $\text{Gal}(K_{/K})$ of $G_{/K}$. Define $\Gamma_K = G_{/K}/H$. Let $\Gamma_0$ denote the subgroup $\text{Gal}(K_{/K}/K_{/K})$ of $\Gamma_K$. Let $\Gamma_i (1 \leq i \leq e)$ be the subgroup of $\Gamma_K$ such that actions of $\Gamma_i$ satisfy $\beta_i(\zeta_{p^m}) = \zeta_{p^m}$ and $\beta_i(b_j^{1/p^m}) = b_j^{1/p^m}$ for $i \neq j$ and define the homomorphism $c_i : \Gamma_i \to \mathbb{Z}_p$ such that we have $\beta_i(b_j^{1/p^m}) = b_j^{1/p^m} \zeta_p^{c_i(\beta_i)}$. Then, the homomorphism $c_i$ defines an isomorphism $\Gamma_i \simeq \mathbb{Z}_p$ of profinite groups. With this, we can see that there exist isomorphisms of profinite groups $\Gamma_K \simeq \Gamma_0 \ltimes \bigoplus_{i=1}^e \Gamma_i \simeq \Gamma_0 \ltimes \mathbb{Z}_p$.

3. Preliminaries on $p$-adic differential modules

In this section, we shall review the theory of $p$-adic differential modules which plays an important role in this article. First, let us fix the notations. Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p > 0$ such that $[k : k^p] = p^c < \infty$ and $V$ be a $p$-adic representation of $G_K$. Define $K_{/K}$ and $K_{/K}$ as in Introduction and Subsection 2.2. Put $K_{/K} = \bigcup_{m \geq 0} K_{/K}(\zeta_{p^m})$ (resp. $K_{/K} = \bigcup_{m \geq 0} K_{/K}(\zeta_{p^m})$) where $\zeta_{p^m}$ denotes a primitive $p^m$-th root of unity in $K_{/K}$ such that $K_{/K}$ is a Hodge-Tate representation of $G_K$. We can show that a potentially Hodge-Tate representation $V$ of $K_{/K}$ is a Hodge-Tate representation of $G_K$ in the same way as in the perfect residue field case.

3.1. Definitions of $p$-adic differential modules. We shall review the definitions of $p$-adic differential modules and have the following diagram, for a $p$-adic representation $V$ of $G_K$,

\[
\begin{array}{ccc}
(B_{\text{dR},K} \otimes_{\mathbb{Q}_p} V)^G_K & \xrightarrow{\theta_K} & (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^G_K \\
\cup & & \cup \\
D^+_\text{dif}(V) & \rightarrow & D_{\text{Sen}}(V) \\
\cup & & \cup \\
D^+_{\text{e-dif}}(V) & \rightarrow & D_{\text{Bri}}(V).
\end{array}
\]
3.1.1. The module $D_{\text{Sen}}(V)$. In the article [S], Sen shows that, for a $p$-adic representation $V$ of $G_{K^p}$, the $K^{\text{pf}}$-vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ has dimension $d = \dim_{\mathbb{Q}_p} V$ and the union of the finite dimensional $K^{\text{pf}}$-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ stable under $\Gamma_0$ ($\simeq G_{K^p}/H$) is a $K^{\text{pf}}$-vector space of dimension $d$ stable under $\Gamma_0$ (called $D_{\text{Sen}}(V)$). We have $\mathbb{C}_p \otimes_{K^{\text{pf}}} D_{\text{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}^{\text{pf}}_\infty \otimes_{K^{\text{pf}}} D_{\text{Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\text{Sen}}(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1 - \gamma)^k}{k}$$

converges to a $K^{\text{pf}}$-linear derivation $\nabla^{(0)} : D_{\text{Sen}}(V) \to D_{\text{Sen}}(V)$ and does not depend on the choice of $\gamma$.

3.1.2. The module $D_{\text{Bri}}(V)$. In the article [Br1], Brinon generalizes Sen’s work above. For a $p$-adic representation $V$ of $G_K$, he shows that the union of the finite dimensional $K^{(p)}_\infty$-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ stable under $\Gamma_K$ is a $K^{(p)}_\infty$-vector space of dimension $d$ stable under $\Gamma_K$ (we call it $D_{\text{Bri}}(V)$). We have $\mathbb{C}_p \otimes_{K^{(p)}_\infty} D_{\text{Bri}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}^{(p)}_\infty \otimes_{K^{(p)}_\infty} D_{\text{Bri}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. As in the case of $D_{\text{Sen}}(V)$, the $K^{(p)}_\infty$-vector space $D_{\text{Bri}}(V)$ is endowed with the action of the $K^{(p)}_\infty$-linear derivation $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D_{\text{Bri}}(V)$

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)} \sum_{k \geq 1} \frac{(1 - \beta_i)^k}{k}$$

converges to a $K^{(p)}_\infty$-linear derivation $\nabla^{(i)} : D_{\text{Bri}}(V) \to D_{\text{Bri}}(V)$ and does not depend on the choice of $\beta_i$.

3.1.3. The module $D^{+}_{\text{ef}}(V)$. In the article [A-B], Andreata and Brinon generalize Fontaine’s work [F3]. For a $p$-adic representation $V$ of $G_K$, they show that the union of $K^{(p)}_\infty[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B^{+}_{\text{ef},K} \otimes_{\mathbb{Q}_p} V)^H$ stable under $\Gamma_K$ is a free $K^{(p)}_\infty[[t, t_1, \ldots, t_e]]$-module of rank $d$ stable under $\Gamma_K$ (we call it $D^{+}_{\text{ef}}(V)$). We have $B^{+}_{\text{ef},K} \otimes_{K^{(p)}_\infty[[t, t_1, \ldots, t_e]]} D^{+}_{\text{ef}}(V) = B^{+}_{\text{ef},K} \otimes_{\mathbb{Q}_p} V$ and the natural map $(B^{+}_{\text{ef},K})^H \otimes_{K^{(p)}_\infty[[t, t_1, \ldots, t_e]]} D^{+}_{\text{ef}}(V) \to (B^{+}_{\text{ef},K} \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism. The $K^{(p)}_\infty[[t, t_1, \ldots, t_e]]$-module $D^{+}_{\text{ef}}(V)$ is endowed with the action of the $K^{(p)}_\infty$-linear derivations $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1 and $\nabla^{(i)} = \frac{\log(\beta_i)}{c_i(\beta_i)} (1 \leq i \leq e)$ if $\beta_i \in \Gamma_i$ is close enough to 1.
3.1.4. The module $D_{\text{e-diff}}^+(V)$. For a $p$-adic representation $V$ of $G_K$, define $D_{\text{e-diff}}^+(V)$ to be $\lim_{\to} (K^p_\infty[[t, t_1, \ldots, t_e]] \otimes_{K^p_\infty[[t, t_1, \ldots, t_e]]} D_{\text{e-diff}}^+(V))$ where we put $D_{\text{e-diff}}^+(V) = D_{\text{e-diff}}^+(V)/\langle t, t_1, \ldots, t_e \rangle D_{\text{e-diff}}^+(V)$. One can verify that $D_{\text{di}}^+(V)$ is the union of $K^p_\infty[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B_{\text{dR}, K} \otimes \mathbb{Q}_p)^H$ stable under $\Gamma_0 (\simeq G_{K^p}/H)$ and is a free $K^p_\infty[[t, t_1, \ldots, t_e]]$-module of rank $d$ stable under $\Gamma_0$. Furthermore, we have $B_{\text{dR}, K} \otimes_{K^p_\infty[[t, t_1, \ldots, t_e]]} D_{\text{di}}^+(V) = B_{\text{dR}, K} \otimes \mathbb{Q}_p$ and the natural map $(B_{\text{dR}, K})^H \otimes_{K^p_\infty[[t, t_1, \ldots, t_e]]} D_{\text{di}}^+(V) \to (B_{\text{dR}, K} \otimes \mathbb{Q}_p)^H$ is an isomorphism. As in the case of $D_{\text{e-diff}}^+(V)$, the $K_{\infty}^p[[t, t_1, \ldots, t_e]]$-module $D_{\text{di}}^+(V)$ is endowed with the action of the $K_{\infty}^p$-linear derivation $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_0$ is close enough to 1.

Remark 3.1. (1) The preceding results in Subsection 3.1.1 are obtained when $V$ is a $p$-adic representation of $G_L = \text{Gal}(\overline{L}/L)$ where $L$ is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and we choose an algebraic closure $\overline{L}$ of $L$. However, in Subsection 3.1.1, for simplicity, we stated the results in the case $L = K_{\infty}^p$.

(2) Note that, though many people denote the $p$-adic differential module constructed by Fontaine in [F3] by $D_{\text{dif}}^+(V)$, the module $D_{\text{dif}}^+(V)$ in Subsection 3.1.4 is a little different from this module.

3.2. Some properties of differential operators. We shall describe the action of derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $D_{\text{dR}}(V)$ and $D_{\text{e-diff}}^+(V)$. First, by a standard argument, we can show that, if $x \in D_{\text{dR}}(V)$ (resp. $D_{\text{e-diff}}^+(V)$), we have

$$\nabla^{(0)}(x) = \lim_{\gamma \to 1} \frac{\gamma(x) - x}{\chi(\gamma) - 1} \quad \text{and} \quad \nabla^{(i)}(x) = \lim_{\beta_i \to 1} \frac{\beta_i(x) - x}{c_i(\beta_i)}.$$

With this, we can easily describe the actions of $K_{\infty}^p$-linear derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $K_{\infty}^p[[t, t_1, \ldots, t_e]] = D_{\text{e-diff}}^+(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is equipped with the structure of $p$-adic representations of $G_K$ induced by the trivial action of $G_K$.

Lemma 3.2. The actions of $K_{\infty}^p$-linear derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $K_{\infty}^p[[t, t_1, \ldots, t_e]]$ are given by $\nabla^{(0)} = \frac{d}{dt}$ and $\nabla^{(i)} = \frac{d}{dt_i}$ ($1 \leq i \leq e$).

Proof. Since $\{\nabla^{(j)}\}_{j=0}^e$ are $K_{\infty}^p$-linear derivations and we can see that we have $\nabla^{(j)}(t_k) = 0$ ($j \neq k$) and $\nabla^{(i)}(t) = 0$ ($i \neq 0$), it suffices to show that we have $\nabla^{(0)}(t) = t$ and $\nabla^{(i)}(t_i) = t_i$. These follow from

$$\nabla^{(0)}(t) = \lim_{\gamma \to 1} \frac{\gamma(t) - t}{\chi(\gamma) - 1} = \lim_{\gamma \to 1} \frac{\chi(\gamma)t - t}{\chi(\gamma) - 1} = t$$

$$\nabla^{(i)}(t_i) = \lim_{\beta_i \to 1} \frac{\beta_i(t_i) - t_i}{c_i(\beta_i)} = \lim_{\beta_i \to 1} \frac{(t_i + c_i(\beta_i)t_i) - t_i}{c_i(\beta_i)} = t.$$

We extend naturally actions of $K_{\infty}^p$-linear derivations $\{\nabla^{(i)}\}_{i=0}^e$ on $K_{\infty}^p[[t, t_1, \ldots, t_e]]$ to $K_{\infty}^p[[t, t_1, \ldots, t_e]][t^{-1}]$ (⊂ $B_{\text{dR}, K}$) by putting $\nabla^{(0)}(t^{-1}) = -t^{-1}$ and
On the For simplicity, put the action of the \( \nabla \) from the equality Proposition 3.4.

\[ \nabla(i)(t^{-1}) = 0 \quad (1 \leq i \leq e). \]

Now, we compute the bracket \([ , ]\) of derivations \(\{\nabla(i)\}_{i=0}^e\) on \(D_{\text{Bri}}(V)\) (resp. \(D_{\text{e-diff}}^+(V)\)).

**Proposition 3.3.** On the \( p \)-adic differential module \(D_{\text{Bri}}(V)\) (resp. \(D_{\text{e-diff}}^+(V)\)), we have \([\nabla(0), \nabla(i)] = \nabla(i) \quad (i \neq 0)\) and \([\nabla(i), \nabla(j)] = 0 \quad (i, j \neq 0)\).

**Proof.** The second equality follows from the commutativity of \(\beta_i\) and \(\beta_j\). For the first equality, we have the relation \(\gamma\beta_i = \beta_i^{\chi(\gamma)}\gamma\). Then, since we have

\[ \lim_{h \to 0} \frac{a^{b+1} - a}{(h + 1) - 1} = \log(a), \]

we obtain

\[ [\nabla(0), \nabla(i)](\ast) = \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi(\gamma) - 1} \frac{\beta_i - 1}{c_i(\beta_i)}(\ast) - \lim_{\beta_i \to 1} \frac{\beta_i - 1}{c_i(\beta_i)} \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi(\gamma) - 1}(\ast) \]

\[ = \lim_{\beta_i \to 1} \lim_{\gamma \to 1} \frac{\beta_i \gamma - \beta_i \gamma}{(\chi(\gamma) - 1)c_i(\beta_i)}(\ast) \]

\[ = \lim_{\beta_i \to 1} \frac{\beta_i \log(\beta_i)}{c_i(\beta_i)}(\ast) \]

\[ = \nabla(i)(\ast). \]

\(\square\)

**Proposition 3.4.** The action of the \(K^{\text{pf}}_{\infty}\)-linear derivation \(\nabla(i) \quad (i \neq 0)\) on \(D_{\text{Bri}}(V)\) is nilpotent.

**Proof.** From the equality \(\nabla(0)\nabla(i) - \nabla(i)\nabla(0) = \nabla(i)\), we get \(\nabla(0)(\nabla(i))r - (\nabla(i))r\nabla(0) = r(\nabla(i))r\) and \(\text{tr}(r(\nabla(i))r) = 0\) for all \(r \in \mathbb{N}\). Since the characteristic of \(K^{\text{pf}}_{\infty}\) is 0, we obtain \(\text{tr}((\nabla(i))r) = 0\) for all \(r \in \mathbb{N}\). As is well known in linear algebra, this shows that the action of the \(K^{\text{pf}}_{\infty}\)-linear derivation \(\nabla(i) \quad (i \neq 0)\) on \(D_{\text{Bri}}(V)\) is nilpotent.

\(\square\)

**Notation.** For simplicity, put

\[ R = K^{\text{pf}}_{\infty}[t, \frac{t_1}{t}, \ldots, \frac{t_e}{t}] \quad \text{or} \quad K^{\text{pf}}_{\infty}[\{t, t_1, \ldots, t_e\}]. \]

**Proposition 3.5.** Let \(M\) be a finitely generated free \(R[1/t]\)-module endowed with \(K^{\text{pf}}_{\infty}\)-linear derivations \(\{\nabla(i)\}_{i=0}^e\) which satisfy the same properties in Lemma 3.2 and Proposition 3.3. Assume that we can choose a basis \(\{g_j\}_{j=1}^d\) of \(M\) over \(R[1/t]\) such that \(\nabla(0)(g_j) = 0\). Then, the action of \(\nabla(i) \quad (i \neq 0)\) on this basis is given by \(\nabla(i)(g_j) = t \sum_{k=1}^d c_k g_k\) where \(c_k\) is an element of \(R\) such that \(\nabla(0)(c_k) = 0\).
Proof. Since \( \{g_j\}_{j=1}^d \) forms a basis of \( M \) over \( R[1/t] \), we can write, for \( i \neq 0 \),

\[
\nabla^{(i)}(g_j) = \sum_{k=1}^d a_k g_k \quad (a_k \in R[1/t]).
\]

Then, the relation \( [\nabla^{(0)}, \nabla^{(i)}] = \nabla^{(i)} (i \neq 0) \) of Proposition 3.3 says that we have

\[
\sum_{k=1}^d \nabla^{(0)}(a_k) g_k = \sum_{k=1}^d a_k g_k.
\]

Note that we have \( \nabla^{(0)}(g_j) = 0 \) by hypothesis. Hence, we obtain the differential equation \( \nabla^{(0)}(a_k) = a_k \). Define an element \( c_k \) of \( R[1/t] \) to be a \( k \)-th element of \( R \). Then, we can see that \( c_k \) satisfies \( \nabla^{(0)}(c_k) = a_k/t - a_k/t = 0 \) and that \( c_k \) is contained in \( R \). Thus, the solution of the differential equation \( \nabla^{(0)}(a_k) = a_k \) in \( R[1/t] \) has the following form

\[
a_k = c_k t
\]

where \( c_k \) is an element of \( R \) such that \( \nabla^{(0)}(c_k) = 0 \). Hence, from (3.1) and (3.2), we obtain, for \( i \neq 0 \), \( \nabla^{(i)}(g_j) = t \sum_{k=1}^d c_k g_k \) where \( c_k \) is an element of \( R \) such that \( \nabla^{(0)}(c_k) = 0 \). \( \Box \)

Corollary 3.6. With notations as in Proposition 3.5 above, we have the following presentation

\[
(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1 + \cdots + k_e} \sum_{k=1}^d c_k g_k
\]

where \( c_k \) is an element of \( R \) such that \( \nabla^{(0)}(c_k) = 0 \).

4. Proof of the main theorem

In this section, we keep the notation and the assumption in Section 3.

4.1. Main theorem for Hodge-Tate representations.

Proposition 4.1. ([S], Section (2.3)) If \( V \) is a Hodge-Tate representation of \( G_{K^p} \), there exists a \( \Gamma_0 \)-equivariant isomorphism of \( \mathbb{K}_\infty^{\text{pf}} \)-vector spaces

\[
D_{\text{Sen}}(V) \cong \bigoplus_{j=1}^{d = \dim_{\mathbb{Q}_p} V} \mathbb{K}_\infty^{\text{pf}}(n_j) \quad (n_j \in \mathbb{Z}).
\]

Remark 4.2. In general, if \( L \) denotes a complete discrete valuation field of characteristic \( 0 \) with perfect residue field of characteristic \( p > 0 \) and \( V \) is a Hodge-Tate representation of \( G_L = \text{Gal}(\overline{L}/L) \) where we choose an algebraic closure \( \overline{L} \) of \( L \), Sen shows that there exists a \( G_L/H \)-equivariant isomorphism of \( L_\infty(= \bigcup_{m \geq 1} L(\zeta_p^m)) \)-vector spaces ([S], Section (2.3))

\[
D_{\text{Sen}}(V) \cong \bigoplus_{j=1}^{d = \dim_{\mathbb{Q}_p} V} L_\infty(n_j) \quad (n_j \in \mathbb{Z}).
\]
Corollary 4.3. For a $p$-adic representation $V$ of $G_K$, assume that $V$ is a Hodge-Tate representation of $G_{K^{pf}}$. Then, there exists a $\nabla^{(0)}$-equivariant isomorphism of $K^{\infty}_{\infty}$-vector spaces

$$D_{\text{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K^{(pf)}_{\infty}(n_j) \quad (n_j \in \mathbb{Z}).$$

Here, $\simeq_{\nabla^{(0)}}$ denotes a $\nabla^{(0)}$-equivariant isomorphism. Furthermore, the multiplicity of $\{n_j\}_{j=1}^{d}$ is the same as that of $\{n_j\}_{j=1}^{d}$ in Proposition 4.1.

Proof. From the presentation of Proposition 4.1, the action of the $K^{\infty}_{\infty}$-linear derivation $\nabla^{(0)}$ on $D_{\text{Sen}}(V)$ is semi-simple and its eigenvalues are integers. Thus, the action of the $K^{(pf)}_{\infty}$-linear derivation $\nabla^{(0)}$ on the subspace $D_{\text{Bri}}(V)$ of $D_{\text{Sen}}(V)$ is also semi-simple and its eigenvalues are the same. Therefore, we obtain a $\nabla^{(0)}$-equivariant isomorphism $D_{\text{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K^{(pf)}_{\infty}(n_j) \quad (n_j \in \mathbb{Z})$. By tensoring $K^{\infty}_{\infty} \otimes K^{\infty}_{\infty}$ over both sides, we obtain $K^{\infty}_{\infty} \otimes K^{\infty}_{\infty} D_{\text{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K^{\infty}_{\infty}(n_j) \quad (n_j \in \mathbb{Z})$. Furthermore, since we have $K^{\infty}_{\infty} \otimes K^{\infty}_{\infty} D_{\text{Bri}}(V) \twoheadrightarrow D_{\text{Sen}}(V)$ by definition and both sides have the same dimension $d$ over $K^{\infty}_{\infty}$, we obtain $K^{\infty}_{\infty} \otimes K^{\infty}_{\infty} D_{\text{Bri}}(V) = D_{\text{Sen}}(V)$ and can see that the multiplicity of $\{n_j\}_{j=1}^{d}$ is the same as that of $\{n_j\}_{j=1}^{d}$ in Proposition 4.1.

Theorem 4.4. Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and $V$ be a $p$-adic representation of $G_K$. Let $K^{pf}$ be the field extension of $K$ defined as before. Then, $V$ is a Hodge-Tate representation of $G_K$ if and only if $V$ is a Hodge-Tate representation of $G_{K^{pf}}$.

Proof. We shall prove the main theorem in two parts.

(1) $V$: HT rep. of $G_K \Rightarrow V$: HT rep. of $G_{K^{pf}}$

Since $V$ is a Hodge-Tate representation of $G_K$, there exists a $G_K$-equivariant isomorphism of $B_{\text{HT},K}$-modules

(4.1) $$B_{\text{HT},K} \otimes_{\mathbb{Q}_p} V \simeq (B_{\text{HT},K})^{d=\dim_{\mathbb{Q}_p} V}.$$

Now, by tensoring $B_{\text{HT},K^{pf}} \otimes B_{\text{HT},K}$ (which is induced by the $G_{K^{pf}}$-equivariant surjection $p : B_{\text{HT},K} \twoheadrightarrow B_{\text{HT},K^{pf}} : t_i/t \mapsto 0$) over (4.1), we obtain a $G_{K^{pf}}$-equivariant isomorphism of $B_{\text{HT},K^{pf}}$-modules

$$B_{\text{HT},K^{pf}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\text{HT},K^{pf}})^{d}.$$

This means that $V$ is a Hodge-Tate representation of $G_{K^{pf}}$. 
(2) $V$: HT rep. of $G_{K'} \Rightarrow V$: HT rep. of $G_K$

For simplicity, put $R = K_\infty^{(p)}[\frac{t_1}{\ell}, \frac{t_2}{\ell}, \ldots, \frac{t_e}{\ell}]$. We shall construct the $K_\infty^{(p)}$-linearly independent elements $\{f_j^{(s)}\}_{j=1}^{d=\dim_{V(\omega)}}$ of $R[1/t] \otimes_{K_\infty^{(p)}} D_{Br}(V)$ (\subset B_{HT,K} \otimes_{\mathbb{Q}_p} V) such that $\nabla^{(i)}(f_j^{(s)}) = 0$ for all $0 \leq i \leq e$ and $1 \leq j \leq d$.

(A) Construction of $\{f_j^{(s)}\}_{j=1}^{d} \in R[1/t] \otimes_{K_\infty^{(p)}} D_{Br}(V)$

From the presentation of Corollary 4.3 above, if we twist by some powers of $t$, we obtain a basis $\{f_j^{(s)}\}_{j=1}^{d} \subset \pi_{B_{Br},V}^{(s)} \otimes_{K_\infty^{(p)}} D_{Br}(V)$ over $R[1/t]$ such that $\nabla^{(0)}(f_j) = 0$ for all $1 \leq j \leq d$. Thus, by applying Corollary 3.6 to the $R[1/t]$-module $R[1/t] \otimes_{K_\infty^{(p)}} D_{Br}(V)$ generated by $\{f_j^{(s)}\}_{j=1}^{d}$, we can deduce

\[(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^{d} c_k f_k\]

where $c_k$ is an element of $R$ such that $\nabla^{(0)}(c_k) = 0$. Furthermore, since the action of $K_\infty^{(p)}$-linear derivation $\nabla^{(i)}$ ($i \neq 0$) on $D_{Br}(V)$ is nilpotent by Proposition 3.4, if we take $n \in \mathbb{N}$ large enough, we obtain

\[(\nabla^{(i)})^n(f_j) = 0 \quad \text{for all} \quad 1 \leq j \leq d \quad \text{and} \quad 1 \leq i \leq e.\]

Define an element $f_j^{(s)}$ of $R[1/t] \otimes_{K_\infty^{(p)}} D_{Br}(V)$ by

\[f_j^{(s)} = \sum_{0 \leq k_1, \ldots, k_e} (-1)^{k_1+\cdots+k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e!} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e} (f_j).\]

Note that this series is a finite sum by (4.3) and thus $f_j^{(s)}$ actually defines an element of $R[1/t] \otimes_{K_\infty^{(p)}} D_{Br}(V)$. Then, it follows easily that we have $\nabla^{(i)}(f_j^{(s)}) = 0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$ by using the Leibniz rule. Furthermore, by using (4.2) and the fact $\nabla^{(0)}(f_j^{(s)}) = 0$, we can deduce that we have $\nabla^{(0)}(f_j^{(s)}) = 0$ for all $1 \leq j \leq d$.

(B) $\{f_j^{(s)}\}_{j=1}^{d} \in R[1/t] \otimes_{K_\infty^{(p)}} D_{Br}(V)$ is linearly independent over $K_\infty^{(p)}$

By the presentation of $f_j^{(s)}$, we have

\[f_j^{(s)} = f_j + g_j \quad (g_j \in (\frac{t_1}{\ell}, \ldots, \frac{t_e}{\ell})(B_{HT,K} \otimes_{\mathbb{Q}_p} V)).\]

Since $\{f_j^{(s)}\}_{j=1}^{d}$ forms a basis of $R[1/t] \otimes_{K_\infty^{(p)}} D_{Br}(V)$ over $R[1/t]$, it is, in particular, linearly independent over $K_\infty^{(p)} (\subset R[1/t])$. Thus, $\{\overline{f}_j^{(s)}\}_{j=1}^{d}$ (denotes the reduction modulo $(t_1, \ldots, t_e)$) is linearly independent over $K_\infty^{(p)}$ and we can see that $\{f_j^{(s)}\}_{j=1}^{d}$ is linearly independent over $K_\infty^{(p)}$ in $R[1/t] \otimes_{K_\infty^{(p)}} D_{Br}(V)$. 

(C) Conclusion

Therefore, on the $K$-vector space generated by $\{f_j^{(i)}\}_{j=1}^d$, $\log(\gamma)$ and $\{\log(\beta_i)\}_{i=1}^e$ act trivially ($\Leftrightarrow \nabla^{(0)}(f_j^{(i)}) = 0$ and $\nabla^{(0)}(f_j^{(i)}) = 0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$). Thus, this means that $\Gamma_K$ acts on this $K$-vector space via finite quotient and there exists a finite field extension $L/K$ in $K_{\infty}^{\text{pf}}$ such that $\{f_j^{(i)}\}_{j=1}^d$ forms a basis of $D_{\text{HT},L}(V)$ over $L$. Since a potentially Hodge-Tate representation of $G_K$ is a Hodge-Tate representation of $G_K$, this completes the proof. □

4.2. Main theorem for de Rham representations.

Lemma 4.5. For a $p$-adic representation $V$ of $G_K$, assume that $V$ is a de Rham representation of $G_{K^{\text{pf}}}$. Then, we can choose a basis $\{h_j\}_{j=1}^d$ of $D^+_{\text{diff}}(V)[1/t]$ over $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]][1/t]$ such that the action of $\Gamma_0$ on $\{h_j\}_{j=1}^d$ is trivial.

Proof. Since $V$ is a de Rham representation of $G_{K^{\text{pf}}}$, there exists a basis $\{h_j\}_{j=1}^d$ of $B_{\text{dr},K^{\text{pf}}} \otimes_{\mathbb{Q}_p} V$ over $B_{\text{dr},K^{\text{pf}}}$ such that the action of $G_{K^{\text{pf}}}$ on $\{h_j\}_{j=1}^d$ is trivial. We can see that these elements $\{h_j\}_{j=1}^d$ are contained in $D^+_{\text{diff}}(V)[1/t]$ by definition. For each $j$, if we twist $h_j$ by some power of $t$, we obtain an element $g_j$ of $B_{\text{dr},K^{\text{pf}}} \otimes_{\mathbb{Q}_p} V$ such that $g_j \notin tB_{\text{dr},K^{\text{pf}}} \otimes_{\mathbb{Q}_p} V$. Then, it follows that $g_j$ is contained in $D^+_{\text{diff}}(V)$ and satisfies $\overline{g_j} \neq 0$ ($-$ denotes the reduction modulo $(t, t_1, \ldots, t_e)D^+_{\text{diff}}(V)$). Since $D^+_{\text{diff}}(V)$ is a free module of rank $d$ over the local ring $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]]$ and $\{\overline{g_j}\}_{j=1}^d$ forms a basis of $D_{\text{Sen}}(V)$ over $K_{\infty}^{\text{pf}}$, the lifting $\{g_j\}_{j=1}^d$ of $\{\overline{g_j}\}_{j=1}^d$ in $D^+_{\text{diff}}(V)$ forms a basis of $D^+_{\text{diff}}(V)$ over $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]]$. Thus, it follows that $\{h_j\}_{j=1}^d$ forms a basis of $D^+_{\text{diff}}(V)[1/t]$ over $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]][1/t]$. □

With notations as above, note that, since we have the inclusion $D^+_{\text{diff}}(V) \hookrightarrow D^+_{\text{diff}}(V)[1/t]$ by definition, any element $g$ of $D^+_{\text{diff}}(V)$ can be written as $g = \sum_{k=1}^{\infty}(\sum_{j=1}^d a_j g_j)^k$ ($a_j \in K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]]$).

Remark 4.6. Keep the notation as in Lemma 4.5. Since we assume that $V$ is a de Rham representation of $G_{K^{\text{pf}}}$, by Corollary 4.3, there exists a basis $\{v_j\}_{j=1}^d$ of $D_{\text{Bri}}(V)$ over $K_{\infty}^{\text{pf}}$ such that $\nabla^{(0)}(v_j) = n_j v_j$. Put $M = \text{Max}(n_j)_{j=1}^d$. Then, for an element $g \in D^+_{\text{diff}}(V)$, there exists an element $\sum_{k=n}^{\infty}(\sum_{j=1}^d c_j g_j)^k$ of $D^+_{\text{diff}}(V)$ such that we can write

$$g = \sum_{k=m}^{M}(\sum_{j=1}^d b_j g_j)^k + \sum_{k=n}^{\infty}(\sum_{j=1}^d c_j g_j)^k$$

$(b_j, c_j \in K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]])$.

Thus, $g' = \sum_{k=m}^{M}(\sum_{j=1}^d b_j g_j)^k$ defines an element of $D^+_{\text{diff}}(V)$.

Lemma 4.7. With notations as above, for an element $g' = \sum_{k=m}^{M}(\sum_{j=1}^d b_j g_j)^k$ of $D^+_{\text{diff}}(V)$, each $(\sum_{j=1}^d b_j g_j)^k$ is contained in $D^+_{\text{diff}}(V)$. 

}\)
We shall prove this lemma by induction on the smallest degree of $g'$ with respect to $t$. Since we have $g' - (\sum_{j=1}^{d} b_{jm} h_j) t^{m} \in D^+_e(V)$ if $(\sum_{j=1}^{d} b_{jm} h_j) t^{m}$ is contained in $D^+_e(V)$, it suffices to show that $(\sum_{j=1}^{d} b_{jm} h_j) t^{m}$ is contained in $D^+_e(V)$. Since the $K^{\text{perf}}_{\infty}[[t_1, \ldots, t_e]]$-linear derivation $\nabla^{(0)}$ acts trivially on $\{h_j\}_{j=1}^{d}$, we have

$$\prod_{k=m+1}^{M} (\nabla^{(0)} - k)(g') = \prod_{k=m+1}^{M} (m - k)(\sum_{j=1}^{d} b_{jm} h_j) t^{m}.$$  

It follows that $(\sum_{j=1}^{d} b_{jm} h_j) t^{m}$ is contained in $D^+_e(V)$ since the action of $\nabla^{(0)}$ on $D^+_e(V)$ is stable. Thus, this completes the proof. \[\square \]

**Proposition 4.8.** For a $p$-adic representation $V$ of $G_K$, assume that $V$ is a de Rham representation of $G_{K^p}$. Then, there exists a $\nabla^{(0)}$-equivariant isomorphism of $K^{(p)}_{\infty}[[t, t_1, \ldots, t_e]]$-modules

$$D^+_e(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K^{(p)}_{\infty}[[t, t_1, \ldots, t_e]](n_j) (n_j \in \mathbb{Z}).$$

\[\text{Proof.}\] Since $V$ is also a Hodge-Tate representation of $G_{K^p}$, by Corollary 4.3, there exists a basis $\{v_j\}_{j=1}^{d}$ of $D^+_e(V)/(t, t_1, \ldots, t_e)D^+_e(V) \simeq D_{\text{Br}}(V)$ over $K^{(p)}_{\infty}$ such that it gives a $\nabla^{(0)}$-equivariant isomorphism of $K^{(p)}_{\infty}$-vector spaces

$$D^+_e(V)/(t, t_1, \ldots, t_e)D^+_e(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K^{(p)}_{\infty}(n_j) : v_j \mapsto t^{n_j}.$$

Since $D^+_e(V)$ is a free module of rank $d$ over the local ring $K^{(p)}_{\infty}[[t, t_1, \ldots, t_e]]$, any lifting $\{g_j\}_{j=1}^{d}$ of $\{v_j\}_{j=1}^{d}$ in $D^+_e(V)$ forms a basis of $D^+_e(V)$ over $K^{(p)}_{\infty}[[t, t_1, \ldots, t_e]]$. Let $\{h_j\}_{j=1}^{d}$ denote a basis of $D^+_e(V)[1/t]$ over $K^{(p)}_{\infty}[[t, t_1, \ldots, t_e]][1/t]$ such that $\nabla^{(0)}(h_j) = 0$ obtained in Lemma 4.5. Then, we may assume that each $g_j$ is written as $g_j = \sum_{k=m}^{M} (\sum_{l=1}^{d} b_{kl} h_l) t^k (b_{kl} \in K^{(p)}_{\infty}[[t_1, \ldots, t_e]])$ where we take $M \in \mathbb{N}$ as in Remark 4.6. Now, define an element $f_j$ of $D^+_e(V)$ (Lemma 4.7 above) by

$$f_j = (\sum_{l=1}^{d} b_{n_l h_l}) t^{n_j}.$$  

It is easy to see $\nabla^{(0)}(f_j) = n_j f_j$. Therefore, the rest is to show that $\{f_j\}_{j=1}^{d}$ forms a basis of $D^+_e(V)$ over $K^{(p)}_{\infty}[[t, t_1, \ldots, t_e]]$. To prove that $\{f_j\}_{j=1}^{d}$ is a lifting of $\{v_j\}_{j=1}^{d}$, it suffices to show $g_j - f_j \in (t, t_1, \ldots, t_e)D^+_e(V)$. For each $g_j$, put $s_k = (\sum_{l=1}^{d} b_{kl} h_l) t^k \in D^+_e(V)$ (Lemma 4.7 above). Since we have $\nabla^{(0)}(s_k) = k s_k$ (denotes the reduction modulo $(t, t_1, \ldots, t_e)$) and this means $s_k$ is an eigenvector of $\nabla^{(0)}$, it follows that the elements $\{v_j, s_k\neq 0\}_{k \neq n_j}$ are linearly independent over $K^{(p)}_{\infty}$ in $D_{\text{Br}}(V)$. Since we have $v_j = \sum_{k=m}^{M} s_k$ by definition,
it follows that we obtain $\frac{s_k}{n_j} = 0$ for $k \neq n_j$. This means that we have $s_k \in (t, t_1, \ldots, t_e)D^+_{e\text{-dif}}(V)$ ($k \neq n_j$) and $g_j - f_j \in (t, t_1, \ldots, t_e)D^+_{e\text{-dif}}(V)$. Thus, this completes the proof. □

**Remark 4.9.** In general, it is evident from the proof that, if $L$ denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$ and $V$ is a de Rham representation of $G_L = \text{Gal}(\overline{L}/L)$ where we choose an algebraic closure $\overline{L}$ of $L$, we have a $\nabla^{(0)}$-equivariant isomorphism of $L_\infty[[t]]$-modules

$$D^+_{\text{dif}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{Q_p} V} L_\infty[[t]](n_j) \quad (n_j \in \mathbb{Z}).$$

**Theorem 4.10.** Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and $V$ be a $p$-adic representation of $G_K$. Let $K^{p^e}$ be the field extension of $K$ defined as before. Then, $V$ is a de Rham representation of $G_K$ if and only if $V$ is a de Rham representation of $G_{K^{p^e}}$.

**Proof.** We shall prove the main theorem in two parts.

(1) **$V$: dR rep. of $G_K \Rightarrow V$: dR rep. of $G_{K^{p^e}}$**

Since $V$ is a de Rham representation of $G_K$, there exists a $G_K$-equivariant isomorphism of $B_{dR,K}$-modules

$$B_{dR,K} \otimes_{Q_p} V \simeq (B_{dR,K})^{d=\dim_{Q_p} V}.$$ (4.4)

Now, by tensoring $B_{dR,K^{p^e}} \otimes_{B_{dR,K}}$ (which is induced by the $G_{K^{p^e}}$-equivariant surjection $p : B_{dR,K} \twoheadrightarrow B_{dR,K^{p^e}} : t_i \mapsto 0$) over (4.4), we obtain a $G_{K^{p^e}}$-equivariant isomorphism of $B_{dR,K^{p^e}}$-modules

$$B_{dR,K^{p^e}} \otimes_{Q_p} V \simeq (B_{dR,K^{p^e}})^d.$$ This means that $V$ is a de Rham representation of $G_{K^{p^e}}$.

(2) **$V$: dR rep. of $G_{K^{p^e}} \Rightarrow V$: dR rep. of $G_K$**

For simplicity, put $R = K^{p^e}_\infty[[t, t_1, \ldots, t_e]]$. We shall construct the $K^{p^e}_\infty$-linearly independent elements $\{f_j^{(s)}\}_{j=1}^{d=\dim_{Q_p} V}$ of $R[1/t] \otimes_{R} D^+_{e\text{-dif}}(V)$ ($\subset B_{dR,K} \otimes_{Q_p} V$) such that $\nabla^{(i)}(f_j^{(s)}) = 0$ for all $0 \leq i \leq e$ and $1 \leq j \leq d$.

**(A) Construction of $\{f_j^{(s)}\}_{j=1}^{d=\dim_{Q_p} V}$**

From the presentation of Proposition 4.8 above, if we twist by some powers of $t$, we obtain a basis $\{f_j\}_{j=1}^{d=\dim_{Q_p} V}$ of $R[1/t] \otimes_{R} D^+_{e\text{-dif}}(V)$ over $R[1/t]$ such that $\nabla^{(0)}(f_j) = 0$ for all $1 \leq j \leq d$. Thus, by applying Corollary 3.6 to the $R[1/t]$-module
\( R[1/t] \otimes_R D_{e\text{-dif}}^+(V) \) generated by \( \{f_j\}_{j=1}^d \), we can deduce
\[
(\nabla(1))^{k_1} \cdots (\nabla(e))^{k_e}(f_j) = \sum_{k=1}^d c_k f_k
\]
where \( c_k \) is an element of \( R \) such that \( \nabla(0)(c_k) = 0 \). Define an element \( f_j^{(s)} \) of \( R[1/t] \otimes_R D_{e\text{-dif}}^+(V) \) by
\[
f_j^{(s)} = \sum_{0 \leq k_1, \ldots, k_e} (-1)^{k_1 + \cdots + k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e!} (\nabla(1))^{k_1} \cdots (\nabla(e))^{k_e}(f_j).
\]
Note that this series converges in \( R[1/t] \otimes_R D_{e\text{-dif}}^+(V) \) for \((t_1, \ldots, t_e)\)-adic topology by (4.5) and thus \( f_j^{(s)} \) actually defines an element of \( R[1/t] \otimes_R D_{e\text{-dif}}^+(V) \). Then, it follows easily that we have \( \nabla(i)(f_j^{(s)}) = 0 \) for all \( 1 \leq i \leq e \) and \( 1 \leq j \leq d \) by using the Leibniz rule. Furthermore, by using (4.5) and the fact \( \nabla(0)(f_j) = 0 \), we can deduce that we have \( \nabla(0)(f_j^{(s)}) = 0 \) for all \( 1 \leq j \leq d \).

(B) \( \{f_j^{(s)}\}_{j=1}^d \in R[1/t] \otimes_R D_{e\text{-dif}}^+(V) \) is linearly independent over \( K^{(p\ell)}_\infty \)

By the presentation of \( f_j^{(s)} \), we have
\[
f_j^{(s)} = f_j + g_j \quad (g_j \in (t_1, \ldots, t_e)(B_{dR,K} \otimes_{\mathbb{Q}_p} V)).
\]
Since \( \{f_j\}_{j=1}^d \) forms a basis of \( R[1/t] \otimes_R D_{e\text{-dif}}^+(V) \) over \( R[1/t] \), it is, in particular, linearly independent over \( K^{(p\ell)}_\infty \) \((\subset R[1/t])\). Thus, \( \{\overline{f}_j = f_j^{(s)}\}_{j=1}^d \) (\( \overline{-} \) denotes the reduction modulo \((t_1, \ldots, t_e)\)) is linearly independent over \( K^{(p\ell)}_\infty \) and we can see that \( \{f_j^{(s)}\}_{j=1}^d \) is linearly independent over \( K^{(p\ell)}_\infty \) in \( R[1/t] \otimes_R D_{e\text{-dif}}^+(V) \).

(C) Conclusion

Therefore, on the \( K \)-vector space generated by \( \{f_j^{(s)}\}_{j=1}^d, \log(\gamma) \) and \( \{\log(\beta_i)\}_{i=1}^e \) act trivially (\( \Leftrightarrow \nabla(0)(f_j^{(s)}) = 0 \) and \( \nabla(i)(f_j^{(s)}) = 0 \) for all \( 1 \leq i \leq e \) and \( 1 \leq j \leq d \)). Thus, this means that \( \Gamma_K \) acts on this \( K \)-vector space via finite quotient and there exists a finite field extension \( L/K \) in \( K^{(p\ell)}_\infty \) such that \( \{f_j^{(s)}\}_{j=1}^d \) forms a basis of \( D_{dR,L}(V) \) over \( L \). Since a potentially de Rham representation of \( G_K \) is a de Rham representation of \( G_K \), this completes the proof.

\[ \square \]

References


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