GENERALIZATION OF THE THEORY OF SEN IN THE SEMI-STABLE REPRESENTATION CASE

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Abstract. For a semi-stable representation $V$, we will construct a subspace $D_{\pi,\text{Sen}}(V)$ of $\mathbb{C}_p \otimes \mathbb{Q}_p V$ endowed with a linear derivation $\nabla^{(\pi)}$. The action of $\nabla^{(\pi)}$ on $D_{\pi,\text{Sen}}(V)$ is closely related to the action of the monodromy operator $N$ on $D_{\text{st}}(V)$. Furthermore, in the geometric case, the action of $\nabla^{(\pi)}$ on $D_{\pi,\text{Sen}}(V)$ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem.

1. Introduction

Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $k$ of characteristic $p > 0$. Choose an algebraic closure $\overline{K}$ of $K$ and consider its $p$-adic completion $\mathbb{C}_p$. By a $p$-adic representation of $G_K = \text{Gal}(\overline{K}/K)$, we mean a finite dimensional vector space $V$ over $\mathbb{Q}_p$ endowed with a continuous action of $G_K$. Put $K_\infty = \bigcup_{0 \leq m} K(\zeta_{p^m})$ where $\zeta_{p^m}$ denote a primitive $p^m$-th root of unity in $K$ satisfying $(\zeta_{p^m} + 1)^p = \zeta_{p^m}$. Let $H_K$ denote the kernel of the cyclotomic character $\chi: G_K \to \mathbb{Z}_p^*$ and define $\Gamma_K$ to be $G_K/H_K \simeq \text{Gal}(K_\infty/K)$. Then, for a $p$-adic representation $V$ of $G_K$, Sen constructs a $K_\infty$-vector space $D_{\text{Sen}}(V)$ of dimension $\dim_{\mathbb{Q}_p} V$ in $(\mathbb{C}_p \otimes \mathbb{Q}_p V)^{H_K}$ equipped with the $K_\infty$-linear derivation $\nabla^{(0)}$ which is the $p$-adic Lie algebra of $\Gamma_K$. In the case when $V$ is a Hodge-Tate representation of $G_K$, the set of eigenvalues of $\nabla^{(0)}$ on $D_{\text{Sen}}(V)$ is exactly the same as the set of Hodge-Tate weights of $V$.

Now, we shall state the aim of this article. First, let us fix some notations. Fix a prime $\pi$ of $\mathcal{O}_K$ (the ring of integers of $K$) and for each $1 \leq m$, fix a $p^m$-th root $\pi^1/p^m$ of $\pi$ in $\overline{K}$ satisfying $(\pi^1/p^{m+1})^p = \pi^1/p^m$. Put $K^{\text{BK}} = \bigcup_{0 \leq m} K(\pi^1/p^m)$ and $K_\infty^{\text{BK}} = \bigcup_{0 \leq m} K(\pi^1/p^m)$. Here, the letter BK stands for the Breuil-Kisin extension. Let $H$ denote the Galois group $\text{Gal}(\overline{K}/K^{\text{BK}}_\infty)$ and define $\Gamma_{\text{BK}}$ to be $\text{Gal}(K^{\text{BK}}_\infty/K_\infty)$. In this article, for a semi-stable representation $V$ of $G_K$, we shall construct a $K^{\text{BK}}_\infty$-vector space $D_{\pi,\text{Sen}}(V)$ of dimension $\dim_{\mathbb{Q}_p} V$ in $(\mathbb{C}_p \otimes \mathbb{Q}_p V)^{H}$ equipped with the $K^{\text{BK}}_\infty$-linear derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$. Here, $\nabla^{(\pi)}$...
denotes the $p$-adic Lie algebra of $\Gamma_{BK}$. Then, the action of $\nabla^{(0)}$ on $D_{n}\cdot\Sen(V)$ tells us about the Hodge-Tate weights as in the case of $D_{\Sen}(V)$ and the action of $\nabla^{(n)}$ on $D_{n}\cdot\Sen(V)$ is closely related to the action of the monodromy operator $N$ on $D_{n}(V)$. Furthermore, in the case $V = H^{n}_{\et}(X \otimes_{K} \mathbb{K}, \mathbb{Q}_{p})$ where $X$ denotes a proper smooth scheme over $K$, the action of $\nabla^{(n)}$ on $D_{n}\cdot\Sen(V)$ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem.

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2. Preliminaries on basic facts

2.1. $p$-adic periods rings and $p$-adic representations. (See [F1] for details.) Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $k$ of characteristic $p > 0$. Put $K_{0} = \text{Frac}(W(k))$ where $W(k)$ denotes the ring of Witt vectors with coefficients in $k$. Choose an algebraic closure $\overline{K}$ of $K$ and consider its $p$-adic completion $\mathbb{C}_{p}$. Put

$$
\widetilde{E} = \lim_{\leftarrow x \rightarrow x^{0}} \mathbb{C}_{p} = \{(x^{(0)}, x^{(1)}, ...) \mid (x^{(i+1)})^{p} = x^{(i)}, x^{(i)} \in \mathbb{C}_{p}\}.
$$

For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\widetilde{E}$, define their sum and product by $(x + y)^{(i)} = \lim_{j \rightarrow \infty} (x^{(i+j)} + y^{(i+j)})^{p^{j}}$ and $(xy)^{(i)} = x^{(i)}y^{(i)}$. Let $\epsilon = (\epsilon^{(n)})$ denote an element of $\widetilde{E}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\widetilde{E}$ is a perfect field of characteristic $p > 0$ and is the completion of an algebraic closure of $k((\epsilon - 1))$ for the valuation defined by $v_{E}(x) = v_{p}(x^{(0)})$ where $v_{p}$ denotes the $p$-adic valuation of $\mathbb{C}_{p}$ normalized by $v_{p}(p) = 1$. The field $\widetilde{E}$ is equipped with an action of a Frobenius $\sigma$ and a continuous action of the Galois group $G_{K} = \text{Gal}(\overline{K}/K)$ with respect to the topology defined by the valuation $v_{E}$. Define $\widetilde{E}^{+}$ to be the ring of integers for this valuation. Put $\widetilde{A}^{+} = W(\widetilde{E}^{+})$ and $\mathbb{B}^{+} = \widetilde{A}^{+}[1/p] = \{\sum_{k \in \mathbb{Z}} p^{k}[x_{k}] \mid x_{k} \in \widetilde{E}^{+}\}$ where $[\ast]$ denotes the Teichmüller lift of $\ast \in \overline{E}$. This ring $\mathbb{B}^{+}$ is equipped with a surjective homomorphism

$$
\theta : \mathbb{B}^{+} \twoheadrightarrow \mathbb{C}_{p} : \sum p^{k}[x_{k}] \mapsto \sum p^{k}x_{k}^{(0)}.
$$

Let $\widetilde{p}$ denote $(p^{(n)}) \in \widetilde{E}^{+}$ such that $p^{(0)} = p$. Then, $\text{Ker}(\theta)$ is the principal ideal generated by $\omega = [\widetilde{p}] - p$. The ring $B^{+}_{\text{dR}}$ is defined to be the $\text{Ker}(\theta)$-adic completion of $\mathbb{B}^{+}$

$$
B^{+}_{\text{dR}} = \lim_{\leftarrow n \geq 0} \mathbb{B}^{+}/(\text{Ker}(\theta)^{n}).
$$

This is a discrete valuation ring and $t = \log([\widetilde{e}])$ which converges in $B^{+}_{\text{dR}}$ is a generator of the maximal ideal. Put $B_{\text{dR}} = B^{+}_{\text{dR}}[1/t]$. The ring $B_{\text{dR}}$ becomes
Thus, for a \( N \)-adic representation \( V \) of \( G_K \) and a filtration defined by \( \text{Fil}^i B_{\text{dR}} = t^i B_{\text{dR}}^+ \) \( (i \in \mathbb{Z}) \). Then, \( (B_{\text{dR}})^G_K \) is canonically isomorphic to \( K \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{dR}}(V) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \) is naturally a \( K \)-vector space. We say that a \( p \)-adic representation \( V \) of \( G_K \) is a de Rham representation of \( G_K \) if we have
\[
\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{dR}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{dR}}(V)).
\]

Define \( B_{\text{HT}} \) to be the associated graded algebra to the filtration \( \text{Fil}^i B_{\text{dR}} \). The quotient \( \text{gr}^i B_{\text{HT}} = \text{Fil}^i B_{\text{dR}} / \text{Fil}^{i+1} B_{\text{dR}} \) \( (i \in \mathbb{Z}) \) is a one-dimensional \( \mathbb{C}_p \)-vector space spanned by the image of \( t^i \). Thus, we obtain the presentation
\[
B_{\text{HT}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)
\]
where \( \mathbb{C}_p(i) = \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i) \) is the Tate twist. Then, \( (B_{\text{HT}})^G_K \) is canonically isomorphic to \( K \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{HT}}(V) = (B_{\text{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K} \) is naturally a \( K \)-vector space. We say that a \( p \)-adic representation \( V \) of \( G_K \) is a Hodge-Tate representation of \( G_K \) if we have
\[
\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{HT}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{HT}}(V)).
\]

Let \( \theta : \mathbb{A}^+ \to \mathcal{O}_{\mathbb{C}_p} \) be the natural homomorphism where \( \mathcal{O}_{\mathbb{C}_p} \) denotes the ring of integers of \( \mathbb{C}_p \). Define the ring \( A_{\text{cris}} \) to be the \( p \)-adic completion of the PD-envelope of \( \text{Ker}(\theta) \) compatible with the canonical PD-envelope over the ideal generated by \( p \). Put \( B_{\text{cris}}^+ = A_{\text{cris}}[1/p] \) and \( B_{\text{cris}} = B_{\text{cris}}^+[1/t] \). These rings are \( K_0 \)-algebras endowed with actions of \( G_K \) and Frobenius \( \varphi \). Furthermore, since these rings are canonically included in \( B_{\text{dR}} \), they are endowed with the filtration induced by that of \( B_{\text{dR}} \). Then, \( (B_{\text{cris}})^G_K \) is canonically isomorphic to \( K_0 \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{cris}}(V) = (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} \) is naturally a \( K_0 \)-vector space. We say that a \( p \)-adic representation \( V \) of \( G_K \) is a crystalline representation of \( G_K \) if we have
\[
\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\text{cris}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\text{cris}}(V)).
\]

Fix a prime element \( \pi \) of \( \mathcal{O}_K \) (the ring of integers of \( K \)) and an element \( s = (s^{(n)}) \in \mathbb{E}^+ \) such that \( s^{(0)} = \pi \). Then, the series \( \log([s] \pi^{-1}) \) converges to an element \( u_\pi \in B_{\text{dR}}^+ \) and the subring \( B_{\text{cris}}[u_\pi] \) of \( B_{\text{dR}} \) depends only on the choice of \( \pi \). We denote this ring by \( B_{\text{st}} \). Since this ring is included in \( B_{\text{dR}} \), it is endowed with the action of \( G_K \) and the filtration induced by those on \( B_{\text{dR}} \). The element \( u_\pi \) is transcendental over \( B_{\text{cris}} \) and we extend the Frobenius \( \varphi \) on \( B_{\text{cris}} \) to \( B_{\text{st}} \) by putting \( \varphi(u_\pi) = p u_\pi \). Furthermore, define the \( B_{\text{cris}} \)-derivation \( B_{\text{st}} \to B_{\text{st}} \) by \( N(u_\pi) = -1 \). It is easy to verify that we have \( N \varphi = p \varphi N \) and that the action of \( N \) on \( D_{\text{st}}(V) \) is nilpotent. As in the case of \( B_{\text{cris}} \), we have \( (B_{\text{st}})^{G_K} = K_0 \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{st}}(V) = (B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} \) is naturally a \( K_0 \)-vector space. We say that a \( p \)-adic representation \( V \) of \( G_K \) is a semi-stable
representation of $G_K$ if we have
\[ \dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{st}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{st}(V)). \]
Furthermore, we say that $V$ is a potentially semi-stable representation of $G_K$ if there exists a finite field extension $L/K$ in $\overline{K}$ such that $V$ is a semi-stable representation of $G_L$. Due to the result of Berger [Be1], it is known that $V$ is a potentially semi-stable representation of $G_K$ if and only if $V$ is a de Rham representation of $G_K$.

2.2. The theory of Sen. Keep the notation and assumption in Introduction. In the article [S3], Sen shows that, for a $p$-adic representation $V$ of $G_K$, the $\hat{K}_\infty(= (\mathbb{C}_p)^{H_K})$-vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ has dimension $d = \dim_{\mathbb{Q}_p} V$ and the union of the finite dimensional $K_\infty$-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under $\Gamma_K$ is a $K_\infty$-vector space of dimension $d$ stable under $\Gamma_K$ (called $D_{Sen}(V)$). We have $\mathbb{C}_p \otimes_{K_\infty} D_{Sen}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}_\infty \otimes_{K_\infty} D_{Sen}(V) \rightarrow (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_K$ is close enough to 1, then the series of operators on $D_{Sen}(V)$
\[ \frac{\log(\gamma)}{\log(\chi(\gamma))} = - \frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1 - \gamma)^k}{k} \]
converges to a $K_\infty$-linear derivation $\nabla^{(0)} : D_{Sen}(V) \rightarrow D_{Sen}(V)$ and does not depend on the choice of $\gamma$. By the following proposition, we can see that the set of eigenvalues of $\nabla^{(0)}$ on $D_{Sen}(V)$ is exactly the same as the set of Hodge-Tate weights of $V$ if $V$ is a Hodge-Tate representation of $G_K$.

**Proposition 2.1.** If $V$ is a Hodge-Tate representation of $G_K$, there exists a $\Gamma_K$-equivariant isomorphism of $K_\infty$-vector spaces
\[ D_{Sen}(V) \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K_\infty(n_j) \quad (n_j \in \mathbb{Z}). \]

**Proof.** Since $V$ is a Hodge-Tate representation of $G_K$, there exists a basis $\{g_j\}_{j=1}^d$ of $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ over $\mathbb{C}_p$ such that it gives the Hodge-Tate decomposition
\[ \mathbb{C}_p \otimes_{\mathbb{Q}_p} V \simeq \bigoplus_{j=1}^d \mathbb{C}_p(n_j) : g_j \mapsto t^{n_j} \quad (n_j \in \mathbb{Z}). \]
From this presentation, it follows that $\{g_j\}_{j=1}^d$ forms a basis of a $K_\infty$-vector space $X$ which is contained in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ and stable under the action of $\Gamma_K$. Then, since we have $X \mapsto D_{Sen}(V)$ by definition and both sides have the same dimension $d$ over $K_\infty$, we get the equality $X = D_{Sen}(V)$. Thus, we obtain the $\Gamma_K$-equivariant isomorphism of $K_\infty$-vector spaces
\[ D_{Sen}(V) \simeq \bigoplus_{j=1}^d K_\infty(n_j) : g_j \mapsto t^{n_j}. \]
Let us recall notations. Fix a prime $\pi$ of $\mathcal{O}_K$ (the ring of integers of $K$) and for each $1 \leq m$, fix $p^m$-th root $\pi^{1/p^m}$ of $\pi$ in $\overline{K}$ satisfying $(\pi^{1/p^m})^p = \pi^{1/p^m}$. Put $K^{BK} = \bigcup_{0 \leq m} K(\pi^{1/p^m})$ and $K^{BK}_\infty = \bigcup_{0 \leq m} K(\pi^{1/p^m})$. Let $H$ denote the Galois group $\text{Gal}(\overline{K}/K^{BK}_\infty)$ and define $\Gamma_{BK}$ to be $\text{Gal}(K^{BK}/K^{BK}_\infty)$. For $\beta \in \Gamma_{BK}$, we have $\beta(\pi_{p^m}) = \pi_{p^m}$ and define the homomorphism $c : \Gamma_{BK} \to \mathbb{Z}_p$ such that we have $\beta(\pi^{1/p^m}) = \pi^{1/p^m} \cdot \pi_{p^m}$. Then, the homomorphism $c$ defines an isomorphism $\Gamma_{BK} \cong \mathbb{Z}_p$ of profinite groups.

**Lemma 3.2.** Let $\{g_j\}$ be the $\mathbb{Z}_p$-linear derivation $\frac{\text{d}}{\text{d}t}$ on $\mathbb{Q}_p^\times$ defined by $\beta(\pi_j) = \pi_j$, where $\pi_j$ denotes the $\mathbb{Z}_p$-valued $j$-th power of $\pi$ in $\mathbb{Q}_p$. For $\beta \in \Gamma_{BK}$, we define $\beta(\pi_j)$ to be $\pi_j^{\beta}$. Then, there exists a basis $\{g_j\}$ of $D_{\text{Sen}}(V)$ over $K_0$ such that we have

\[ g_1 \rightarrow g_2 \rightarrow \cdots \rightarrow g_d \rightarrow 0. \]

By twisting $\{g_j\}$ by some powers of $t$ in $B_{st} \otimes_{\mathbb{Q}_p} V$, we obtain a basis $\{f_j\}$ of $B_{\text{cris}}^+[\pi/t] \otimes_{\mathbb{Q}_p} V$ over $B_{\text{cris}}^+[\pi/t]$. Then, we can write

\[
\begin{align*}
\circ f_1 &= t^{m_1(=0)}(F_1 + (-1)^{1} \frac{1}{1!} \pi_2 F_2 + (-1)^{2} \frac{2}{2!} \pi_3 F_3 + \cdots + (-1)^{d-1} \frac{u_{d-1}^{d-1}}{(d-1)!} \pi_d F_d) \\
\circ f_2 &= t^{m_2}(F_2 + (-1)^{1} \frac{1}{1!} \pi_2 F_3 + \cdots + (-1)^{d-2} \frac{u_{d-2}^{d-2}}{(d-2)!} \pi_d F_d) \\
&\vdots \\
\circ f_d &= t^{m_d} F_d
\end{align*}
\]

where $\{F_j\}$ denotes a set of elements of $B_{\text{cris}}^+ \otimes_{\mathbb{Q}_p} V$ and we take $m_j \in \mathbb{Z}$ such that $\{f_j\}$ forms a basis of $B_{\text{cris}}^+[\pi/t] \otimes_{\mathbb{Q}_p} V$ over $B_{\text{cris}}^+[\pi/t]$.

**Definition 3.1.** With notations as above, let $\{h_j := t^{m_j} F_j\}$ denote the image of $\{t^{m_j} F_j\}$ by the homomorphism $B_{\text{cris}}^+ \otimes_{\mathbb{Q}_p} V \to \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. Then, define $D_{\text{Sen}}(V)$ to be the $K_{BK}$-vector space generated by $\{h_j\}$ contained in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$.

**Lemma 3.2.** The elements $\{h_j\}$ are linearly independent over $\mathbb{C}_p$ in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. In particular, $\{h_j\}$ forms a basis of $D_{\text{Sen}}(V)$ over $K_{BK}$ and its dimension over $K_{BK}$ is equal to $\dim_{\mathbb{Q}_p} V$.

**Proof.** We can show inductively that $\{h_d\}, \{h_{d-1}, h_d\}, \ldots, \{h_1, \ldots, h_d\}$ are linearly independent over $\mathbb{C}_p$ in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. By this lemma, we can easily verify that the following proposition holds.

**Proposition 3.3.** (c.f. Subsection 2.2) We have $\mathbb{C}_p \otimes_{K_{BK}} D_{\text{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $(\mathbb{C}_p)^H \otimes_{K_{BK}} D_{\text{Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ is an isomorphism.

It follows easily that the $K_{BK}$-vector space $D_{\text{Sen}}(V)$ is equipped with the $K_{BK}$-linear derivation $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_K$ is close enough to 1. By the
following proposition, we can see that the action of $\nabla^{(0)}$ on $D_{\pi\text{-Sen}}(V)$ tells us about Hodge-Tate weights as in the case of $D_{\text{Sen}}(V)$.

**Proposition 3.4.** (c.f. Proposition 2.1) For a semi-stable representation $V$ of $G_K$, there exists a $\Gamma_K$-equivariant isomorphism of $K^{\text{BK}}_{\infty}$-vector spaces

$$D_{\pi\text{-Sen}}(V) \simeq \bigoplus_{j=1}^{d} K^{\text{BK}}_{\infty}(n_j) : h_j \mapsto t^{n_j} \ (n_j \in \mathbb{Z}).$$

Furthermore, the set of integers $\{n_j\}_j$ is exactly the same as the set of Hodge-Tate weights of $V$.

**Proof.** Note that we have $\{\gamma(f_j) = \chi(\gamma)^{n_j} f_j\}_j$ by definition. Then, we can show inductively that we have $\{\gamma(h_d) = \chi(\gamma)^{n_d} h_d\}$, $\{\gamma(h_{d-1}) = \chi(\gamma)^{n_{d-1}} h_{d-1}\}$, ..., $\{\gamma(h_1) = \chi(\gamma)^{n_1} h_1\}$. The rest is easily verified by Proposition 3.3. \qed

On the other hand, if $\beta \in \Gamma_{\text{BK}}$ is close enough to 1, the series of operators on $D_{\pi\text{-Sen}}(V)$

$$\nabla^{(\pi)} = \frac{\log(\beta)}{c(\beta)} = -\frac{1}{c(\beta)} \sum_{k \geq 1} \frac{(1-\beta)^k}{k}$$

converges to a $K^{\text{BK}}_{\infty}$-linear derivation on $D_{\pi\text{-Sen}}(V)$ does not depend on the choice of $\beta \in \Gamma_{\text{BK}}$. This easily follows from the calculations $\nabla^{(\pi)}(f_j) = 0$ and $\nabla^{(\pi)}(\frac{u_\pi}{T}) = 1$.

**Remark 3.5.** By using the calculations $\nabla^{(\pi)}(f_j) = 0$ and $\nabla^{(\pi)}(\frac{u_\pi}{T}) = 1$, we obtain $\nabla^{(\pi)}(F_j) = F_{j+1} (j < d)$ and $\nabla^{(\pi)}(F_d) = 0$. Thus, we can rewrite ($\ast$) as

$$(\ast)' \quad f_j = t^{m_j} \left(\sum_{k=0}^{d-j} (-1)^k \frac{u_\pi^k}{k! k^k} (\nabla^{(\pi)})^k(F_j)\right) \quad (1 \leq j \leq d).$$

Compare this formula to the main construction $\{f^{(\ast)}_j\}_j$ in [M1]. In fact, the idea of the construction of $D_{\pi\text{-Sen}}(V)$ is based on the similarity between Corollary 2.1.14 of [Ki] and Main Theorems of [M1] and [M2].

### 3.2. Some properties of differential operators

We shall describe the actions of derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$. First, by a standard argument, we can show that, if $x \in D_{\pi\text{-Sen}}(V)$, we have

$$\nabla^{(0)}(x) = \lim_{\gamma \to 1} \frac{\gamma(x) - x}{\chi(\gamma) - 1} \quad \text{and} \quad \nabla^{(\pi)}(x) = \lim_{\beta \to 1} \frac{\beta(x) - x}{c(\beta)}.$$

By using these presentations, we compute the bracket $[\ , \ ]$ of derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$ on $D_{\pi\text{-Sen}}(V)$.

**Proposition 3.6.** On the differential module $D_{\pi\text{-Sen}}(V)$, we have $[\nabla^{(0)}, \nabla^{(\pi)}] = \nabla^{(\pi)}$. 
Proof. First, note that we have the relation $\gamma \beta = \beta^{\chi(\gamma)} \gamma$. Then, since we have
$$\lim_{h \to 0} \frac{a^{h+1} - a}{(h + 1) - 1} = a \log(a),$$
we obtain
$$[\nabla^{(0)}, \nabla^{(\pi)}](*) = \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi(\gamma) - 1} \lim_{\beta \to 1} \frac{\beta - 1}{c(\beta)}(*) - \lim_{\beta \to 1} \frac{\beta - 1}{c(\beta)} \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi(\gamma) - 1} (*)$$
$$= \lim_{\beta \to 1} \frac{\beta^{\chi(\gamma)} \gamma - \beta^\gamma}{(\chi(\gamma) - 1)c(\beta)}(*)$$
$$= \lim_{\beta \to 1} \frac{\beta \log(\beta)}{c(\beta)}(*)$$
$$= \nabla^{(\pi)}(*).$$

Proposition 3.7. The action of the $K^\infty$-linear derivation $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is nilpotent.

Proof. From the equality $\nabla^{(0)} \nabla^{(\pi)} = \nabla^{(\pi)} \nabla^{(0)} = \nabla^{(\pi)}$, we get $\nabla^{(0)}(\nabla^{(\pi)})^r = (\nabla^{(\pi)})^r\nabla^{(0)} = r(\nabla^{(\pi)})^r$ and $\text{tr}(r(\nabla^{(\pi)})^r) = 0$ for all $r \in \mathbb{N}$. Since the characteristic of $K^\infty$ is 0, we obtain $\text{tr}(r(\nabla^{(\pi)})^r) = 0$ for all $r \in \mathbb{N}$. As is well known in linear algebra, this shows that the action of the $K^\infty$-linear derivation $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is nilpotent. □

Proposition 3.8. For an element $x \in D_{\pi-Sen}(V)$ such that $\nabla^{(0)}(x) = nx$ ($n \in \mathbb{Z}$), we have $\nabla^{(0)}(\nabla^{(\pi)}(x)) = (n + 1)\nabla^{(\pi)}(x)$, that is, the action of $\nabla^{(\pi)}$ increases the Hodge-Tate weight by 1.

Proof. This follows easily from the relation $[\nabla^{(0)}, \nabla^{(\pi)}] = \nabla^{(\pi)}$. □

There are many choices of a $K^\infty$-subspace of dimension $\dim_{\mathbb{Q}_p} V$ in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ equipped with derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$. The aim of this article is, however, to construct a differential module in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ which is closely related to the module $D_{\pi}(V)$. Thus, the following proposition says that the choice $D_{\pi-Sen}(V)$ may be a reasonable one.

Proposition 3.9. For a crystalline representation $V$ of $G_K$, the action of $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is trivial.

Proof. In the case when $V$ is a crystalline representation of $G_K$, we can take $\{f_j\}_j$ as a basis of $D_{\pi-Sen}(V)$ over $K^\infty$. We can see that the action of $\Gamma_{BK}$ on this basis is trivial and thus the action of $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is trivial. □
Conversely, there is a semi-stable representation $V$ of $G_K$ such that the action of $\nabla^{(\pi)}$ on $D_{\pi, \Sen}(V)$ is non-trivial. The next example is the prototype of such a semi-stable representation.

**Example 3.10.** Let $V$ be a $p$-adic representation of $G_K$ attached to the Tate curve $\mathbb{K}^*/\langle \pi \rangle$. We can take a basis $\{e, f\}$ of $V$ over $\mathbb{Q}_p$ such that the action of $g \in G_K$ is given by

$$(\chi(g) \ c(g)) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

It is easy to see that $\{h_1 = 1 \otimes f, h_2 = 1 \otimes e\} (\subset C_p \otimes_{\mathbb{Q}_p} V)$ forms a basis of $D_{\pi, \Sen}(V)$ over $K_{\infty}^{BK}$. As indicated by Proposition 3.4, we have

$$\nabla^{(0)}(h_1) = 0 \quad \text{and} \quad \nabla^{(0)}(h_2) = h_2,$$ 

that is, the Hodge-Tate weights of $V$ are $\{0, 1\}$. Furthermore, the action of $\nabla^{(\pi)}$ on this basis is given by

$$h_1 \xrightarrow{\nabla^{(\pi)}} h_2 \xrightarrow{\nabla^{(\pi)}} 0.$$ 

This means that the action of $\nabla^{(\pi)}$ on $D_{\pi, \Sen}(V)$ is nilpotent (Proposition 3.7) and that the action of $\nabla^{(\pi)}$ increases the Hodge-Tate weights by 1 (Proposition 3.8). Thus, we can know more than Hodge-Tate weights by using the $K_{\infty}^{BK}$-vector space $D_{\pi, \Sen}(V)$ equipped with $\nabla^{(\pi)}$.

**4. Geometric aspect of $D_{\pi, \Sen}(V)$**

Let $X$ be a proper smooth scheme over $K$. Then, it is known that the $p$-adic étale cohomology $V^m = H^m_{\et}(X \otimes K, \mathbb{Q}_p)$ is a de Rham representation of $G_K$. Furthermore, due to the result of Berger, we can see that $V^m$ is a potentially semi-stable representation of $G_K$. Let $L/K$ be a finite field extension of $K$ in $\mathbb{K}$ such that $V^m$ is a semi-stable representation of $G_L$ and let $V^m_L$ denote the restriction of $V^m$ to $G_L$.

In this section, we shall study the geometric aspect of $D_{\pi, \Sen}(V^m_L)$ and see that the action of $\nabla^{(\pi)}$ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem. First, by Proposition 3.4, we obtain the $\Gamma_K$-equivariant isomorphism of $L_{\infty}^{BK}$-vector spaces $D_{\pi, \Sen}(V^m_L) \cong \bigoplus_{J=1}^{\dim_{\mathbb{Q}_p} V} L_{\infty}^{BK}(n_J)$.

With this presentation, define the subspace $D_{\pi, \Sen}^{s,t}(V^m_L)$ of $D_{\pi, \Sen}(V^m_L)$ to be $D_{\pi, \Sen}^{m-t,s}(V^m_L) = \{x \in D_{\pi, \Sen}(V^m_L) | \nabla^{(0)}(x) = tx\}$ ($t \in \mathbb{Z}$). It follows easily that we obtain the decomposition

$$D_{\pi, \Sen}(V^m_L) = \bigoplus_{s+t=m} D_{\pi, \Sen}^{s,t}(V^m_L).$$

The next proposition claims that the action of $\nabla^{(\pi)}$ on $D_{\pi, \Sen}(V^m_L)$ satisfies a formula similar to Griffiths transversality.
Proposition 4.1. (Transversality) With notations as above, we have
\[
\nabla^{(\pi)}(D_{\pi-Sen}^{s,t}(V^m_L)) \subset D_{\pi-Sen}^{s-1,t+1}(V^m_L).
\]

Proof. This follows easily from Proposition 3.8.  

By the same argument, we can see that an analogy of the local monodromy theorem holds for the \(L^{\infty}_{BK}\)-vector space \(D_{\pi-Sen}(V^m_L)\) equipped with \(\nabla^{(\pi)}\).

Proposition 4.2. (Local monodromy theorem) With notations as above, the \(L^{\infty}_{BK}\)-linear operator \(\nabla^{(\pi)}\) satisfies
\[
(\nabla^{(\pi)})^{m+1} | D_{\pi-Sen}(V^m_L) = 0.
\]

Furthermore, if we put \(h^{s,t} = \dim_{L^{\infty}_{BK}} D_{\pi-Sen}^{s,t}(V^m_L)\) and define \(h_m = \sup \{b - a | \forall i \in [a, b], h^{i,m-i} \neq 0\}\), we have
\[
(\nabla^{(\pi)})^{h_m+1} | D_{\pi-Sen}(V^m_L) = 0.
\]

References